QUIZ Solution - CALCULUS 1

Q1 :

1. Simplify following expressions.

(a) $\dfrac{(4x)^{\frac{2}{3}}(\frac{1}{2}y^{-2})^{-\frac{1}{3}}}{\sqrt[3]{xy^4}}$.

(b) $\sin^{-1}\left(\sin\left(\dfrac{2}{3}\pi\right)\right)$.

(c) $\tan\left(\cos^{-1}\left(\dfrac{1}{4}\right)\right)$.

sol:

(a) $\dfrac{(4x)^{\frac{2}{3}}(\frac{1}{2}y^{-2})^{-\frac{1}{3}}}{\sqrt[3]{xy^4}} = \dfrac{2^{\frac{4}{3}} \cdot x^{\frac{2}{3}} \cdot 2^{\frac{1}{3}} y^{\frac{2}{3}}}{x^{\frac{1}{3}} \cdot y^{\frac{4}{3}}} = 2^{\frac{5}{3}} x^{\frac{1}{3}} y^{-\frac{2}{3}}$

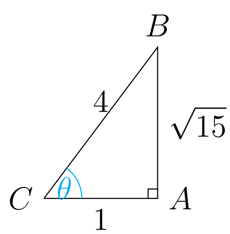
(b) $\sin^{-1}\left(\sin\left(\dfrac{2}{3}\pi\right)\right) = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$

(c) **Solution 1.**

Let $\theta = \cos^{-1}\left(\dfrac{1}{4}\right)$ i.e. $\cos\theta = \dfrac{1}{4}$ and $\theta \in [0, \pi]$.

$\because \cos\theta > 0 \therefore \theta \in [0, \dfrac{\pi}{2})$

Draw a right triangle with an angle θ.



Hence $\tan\theta = \dfrac{\sqrt{15}}{1} = \sqrt{15}$.

Solution 2.

Let $\theta = \cos^{-1}\left(\dfrac{1}{4}\right)$ i.e. $\cos\theta = \dfrac{1}{4}$ and $\theta \in [0, \pi]$.

Hence $\sin^2\theta = 1 - \cos^2\theta = 1 - \dfrac{1}{16}$, $\sin\theta = \pm\dfrac{\sqrt{15}}{4}$

$\because \theta \in [0, \pi] \therefore \sin\theta \geq 0$

Thus $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \sqrt{15}$

Solution 3.

Let $\theta = \cos^{-1}\left(\frac{1}{4}\right)$ i.e. $\cos\theta = \frac{1}{4}$ and $\theta \in [0, \pi]$.

$\because \cos\theta = \frac{1}{4} > 0 \therefore \theta \in [0, \frac{\pi}{2})$.

$\sec\theta = \frac{1}{\cos\theta} = 4$. $\tan^2\theta = \sec^2\theta - 1 = 15$

$\Rightarrow \tan\theta = \pm\sqrt{15}$

$\because \theta \in [0, \frac{\pi}{2}) \therefore \tan\theta > 0$

Hence $\tan\theta = \sqrt{15}$

2. Consider the function $f(x) = \log_4(2x - 6)$.

(a) Use the laws of logarithms and change of base formula to express $f(x)$ as $a + b\ln(x + c)$. Find constants a, b, and c.

(b) (continued) Sketch the graphs of $\ln x$, $\ln(x + c)$, $b\ln(x + c)$, and $f(x) = a + b\ln(x + c)$.

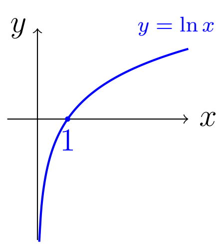
sol:

(a) $f(x) = \log_4(2x - 6) = \log_4 2 + \log_4(x - 3) = \frac{1}{2} + \frac{\ln(x-3)}{\ln 4} = a + b\ln(x + c)$

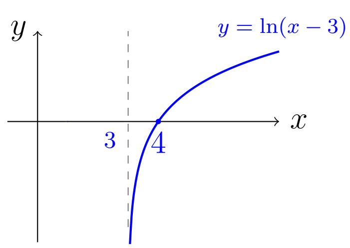
This means that $a = \frac{1}{2}$, $b = \frac{1}{\ln 4} = \frac{1}{2\ln 2}$, $c = -3$

(b)

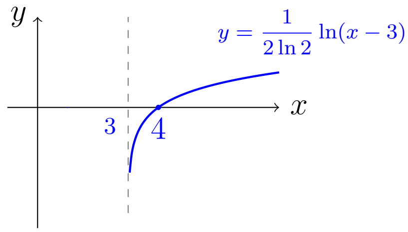
(i)



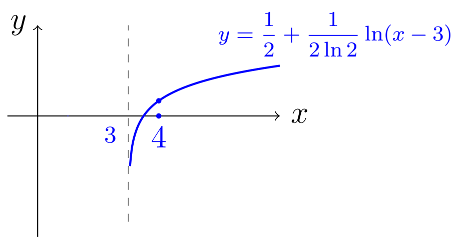
(ii) $y = \ln x \quad 3$



(iii) $y = \ln(x-3) \quad 2\ln 2$



(iv) $y = \dfrac{1}{2\ln 2}\ln(x-3) \quad \dfrac{1}{2}$



Q2 :

1. Evaluate the limits. If the limit does not exist, determine whether the limit is ∞, $-\infty$, or neither. (You CANNOT use any method that uses the derivatives)

(a) $\lim\limits_{x\to 0} \dfrac{(x-3)^2-9}{x^2+2x}$.

Solution:

Factor both the numerator and the denominator.

$$\frac{(x-3)^2-9}{x^2+2x} = \frac{x^2-6x+9-9}{x^2+2x} = \frac{x(x-6)}{x(x+2)}.$$

When $x \neq 0$,

$$\frac{(x-3)^2-9}{x^2+2x} = \frac{x-6}{x+2}.$$

Therefore

$$\lim_{x\to 0} \frac{(x-3)^2-9}{x^2+2x} = \lim_{x\to 0} \frac{x-6}{x+2} = -3$$

□

(b) $\lim\limits_{x\to -4^-} \dfrac{e^x}{x+4}$.

Solution:

Because $\lim\limits_{x\to -4^-} e^x = e^{-4}$ and $\lim\limits_{x\to -4^-} x+4 = 0$, the limit does not exist.

The limit of the numerator is a positive number and the limit of the denominator is 0^-. The infinite limit

$$\lim_{x\to -4^-} \frac{e^x}{x+4} = -\infty$$

$\square$

(c) $\lim_{x\to 0^+} (\sin x)\left(\sin\frac{1}{x}\right)$.

Solution:

Because $\lim_{x\to 0^+} \sin x = 0$ and $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$ does not exist, we need to use other methods to find the limit.

When $x > 0$, we can use the inequalities

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1,$$

$$-|\sin x| \le (\sin x)\left(\sin\frac{1}{x}\right) \le |\sin x|.$$

Since $\lim_{x\to 0^+} -|\sin x| = 0$ and $\lim_{x\to 0^+} |\sin x| = 0$, we can use the Squeeze Theorem. Therefore

$$\lim_{x\to 0^+} (\sin x)\left(\sin\frac{1}{x}\right) = 0$$

$\square$

(d) $\lim_{x\to\infty} \sqrt{3x + 4x^2} - 2x$.

Solution:

This one is straightforward.

$$\lim_{x\to\infty} \sqrt{3x + 4x^2} - 2x$$

$$= \lim_{x\to\infty} \frac{3x + 4x^2 - (2x)^2}{\sqrt{3x + 4x^2} + 2x} = \lim_{x\to\infty} \frac{3x}{\sqrt{3x + 4x^2} + 2x} = \lim_{x\to\infty} \frac{3}{\sqrt{\frac{3}{x} + 4} + 2} = \frac{3}{4}$$

$\square$

(e) $\lim_{x\to 2} g(x)$, where $g(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2, \\ 3, & x = 2, \\ xe^{x-2}, & x > 2. \end{cases}$

Solution:

Check one-sided limits for piecewise functions.

$$\lim_{x\to 2^-} \frac{x^2-4}{x-2} = 4$$

and

$$\lim_{x\to 2^+} xe^{x-2} = 2.$$

Therefore the limit does not exist. $\square$

2. Find the domain of the function $f(x) = \dfrac{x}{\sqrt{x^2 - 3x + 2}}$. Then find the horizontal and vertical asymptotes of the curve $y = f(x)$.

Solution:

The domain is determined by $\sqrt{x^2 - 3x + 2} \neq 0$ and $x^2 - 3x + 2 \geq 0$. Solving them will give us $x \neq 1, 2$ and either $x < 1$ or $x > 2$. In interval notation the domain is $(-\infty, 1) \cup (2, \infty)$.

The function is continuous on its domain. To find horizontal and vertical asymptotes we need to evaluate four limits:

$$\lim_{x \to -\infty} f(x), \lim_{x \to 1^-} f(x), \lim_{x \to 2^+} f(x), \lim_{x \to \infty} f(x).$$

We find

$$\lim_{x \to -\infty} f(x) = -1, \lim_{x \to 1^-} f(x) = \infty, \lim_{x \to 2^+} f(x) = \infty, \lim_{x \to \infty} f(x) = 1,$$

so the asymptotes of $y = f(x)$ are $y = -1, y = 1, x = 1$, and $x = 2$. □

Q3 :

1. Compute $f'(x)$ where

$$f(x) = \frac{\sin x + x^3 - 2x + 5}{2x^2 + 31} - e^x (x + 3) \cos x.$$

Solution:

$$\begin{aligned} f'(x) =& \frac{(\cos x + 3x^2 - 2)(2x^2 + 31) - (\sin x + x^3 - 2x + 5)4x}{(2x^2 + 31)^2} \\ & - (e^x (x + 3) \cos x + e^x (\cos x - (x + 3) \sin x)) \\ =& \frac{(2x^2 + 31) \cos x - 4x \sin x + 2x^4 + 97x^2 - 62}{(2x^2 + 31)^2} - e^x \left((x + 4) \cos x - (x + 3) \sin x \right). \end{aligned}$$

.

2. This is an alternate way to compute $(\ln x)'$.

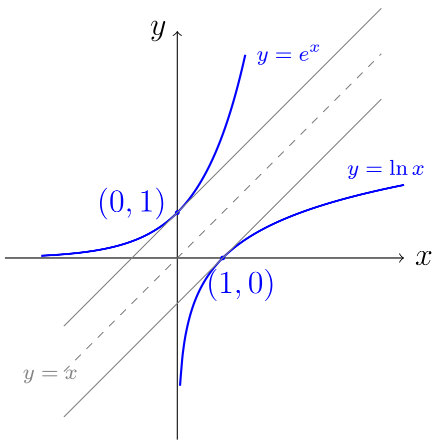
(a) Sketch the graph of $y = e^x$ and $y = \ln x$ on the same picture. Draw the tangent line of $y = e^x$ at $(0, 1)$ and the tangent line of $y = \ln x$ at $(1, 0)$. What are the slopes of those tangents?

(b) Use the result of (a) to show that $\lim_{r \to 1} \dfrac{\ln r}{r - 1} = 1$.

(c) Use the result of (b) to compute $(\frac{d}{dx} \ln x)|_{x=a} = \lim_{x \to a} \dfrac{\ln x - \ln a}{x - a}$. (Hint: Use the Laws of Logarithms and the fact that $\dfrac{x}{a} \to 1$ as $x \to a$.)

Solution:

(a)



The slope of the tangent line of $y = e^x$ at $(0, 1)$ is 1 , since the graph of $y = \ln x$ is a reflection of the graph of $y = e^x$ with respect to $y = x$, the slope of the tangent line of $y = \ln x$ at $(1, 0)$ is also 1.

(b)

$$\lim_{r\to 1} \frac{\ln r}{r-1} = \lim_{r\to 1} \frac{\ln r - \ln 1}{r-1} = (\ln x)'(1)$$

is the slope of the tangent of the graph $y = \ln x$ at $(1, 0)$. Hence $\lim_{r\to 1} \dfrac{\ln r}{r-1} = 1$.

(c)

$$\begin{aligned}\left(\frac{d}{dx}\ln x\right)\Big|_{x=a} &= \lim_{x\to a}\frac{\ln x - \ln a}{x-a} \\ &= \lim_{x\to a}\frac{\ln(\frac{x}{a})}{a(\frac{x}{a}-1)} \\ &\overset{\text{let } r=\frac{x}{a}}{=\!=\!=} \frac{1}{a}\lim_{r\to 1}\frac{\ln r}{r-1} \\ &= \frac{1}{a}\end{aligned}$$

Q4 :

1. Let $f(x) = \ln|\tan^{-1} x|$, $x \neq 0$. Find the equation of the tangent line to the graph of f at $x = -1$.

   Solution:

   Recall that $(\ln|x|)' = \dfrac{1}{x}$ for $0 \neq x \in \mathbb{R}$. Therefore, by the chain rule, we have

   $$f'(-1) = \frac{1}{\tan^{-1} x}\cdot\frac{1}{1+x^2}\Big|_{x=-1} = -\frac{4}{\pi}\cdot\frac{1}{2} = -\frac{2}{\pi}.$$

   So the equation is

   $$y - f(-1) = y - \ln(\frac{\pi}{4}) = -\frac{2}{\pi}(x+1).$$

2. Let the curve in the plane be described by the equation

$$\cos(xy) - \frac{\sqrt{2}}{\pi}x = 0. \tag{1}$$

Find the slope of the tangent line to this curve at $(\pi/2, 1/2)$ by

(a) first solving the function $y = f(x)$ from (1) (*you need to specify the domain of* f) and then taking derivative;

Solution:
We can solve for y to obtain

$$y(x) = \frac{1}{x}\cos^{-1}(\frac{\sqrt{2}}{\pi}x), \quad |x| \le \frac{\pi}{\sqrt{2}}, x \ne 0.$$

Note that $\cos^{-1}(\frac{\sqrt{2}}{\pi}\frac{\pi}{2}) = \frac{\pi}{4}$ and

$$(\cos^{-1}(\frac{\sqrt{2}}{\pi}x))'|_{x=\pi/2} = -\frac{1}{\sqrt{1-1/2}} \cdot \frac{\sqrt{2}}{\pi} = -\frac{2}{\pi}.$$

So the slope is

$$f'(\frac{\pi}{2}) = \frac{x(\cos^{-1}(\frac{\sqrt{2}}{\pi}x))' - \cos^{-1}(\frac{\sqrt{2}}{\pi}x)}{x^2}\Big|_{x=\pi/2} = \frac{-4-\pi}{\pi^2}.$$

(b) implicit differentiation.

Solution:
By the implicit differentiation, we obtain

$$-\sin(xy)[y + xy'] - \frac{\sqrt{2}}{\pi} = 0.$$

So we can evaluate y' at $x = \pi/2$ from

$$-\sin(\frac{\pi}{2}\cdot\frac{1}{2})[\frac{1}{2} + \frac{\pi}{2}y'] = \frac{\sqrt{2}}{\pi}$$

and obtain

$$y'(\frac{\pi}{2}) = \frac{-4-\pi}{\pi^2}.$$

3. Let $f(x) = x^x$ for $x > 0$. Use the linear approximation to approximate the value of $f(1.01)$.

Solution:
We first find $f'(x)$, that is,

$$f'(x) = (x^x)' = (e^{\ln x^x})' = (e^{x\ln x})' = e^{x\ln x}[\ln x + x \cdot \frac{1}{x}] = x^x(\ln x + 1)$$

and so $f'(1) = 1$. Thus, the linear approximation is

$$f'(1.01) \approx f(1) + f'(1)(0.01) = 1.01.$$

Q5 :

1. True or False Questions. Write "T" before correct statements. Write "F" before incorrect statements.

- _______ If $f(c)$ is a local extreme value, then $f'(c) = 0$.
- _______ If c is a critical number of $f(x)$, then $f(c)$ must be a local extreme value.
- _______ If $f(x)$ is differentiable and increasing on an interval I, then $f'(x) > 0$ for all $x \in I$.
- _______ If $f''(c) = 0$, then $(c, f(c))$ is an inflection point of $y = f(x)$.

  Solution: Statements in problem 1 are all False.

2. Consider the function $f(x) = -2x^2 + 5x - \ln x$.

- Compute $f'(x)$. Find intervals on which f is increasing or decreasing.

  Solution:
  $f'(x) = -4x + 5 - \dfrac{1}{x}$.
  $f'(x) = \dfrac{-1}{x}(4x-1)(x-1)$.
  Hence $f'(x) < 0$ for $x \in (0, \frac{1}{4}) \cup (1, \infty)$ and $f'(x) > 0$ for $x \in (\frac{1}{4}, 1)$. Therefore $f(x)$ is increasing on $(\frac{1}{4}, 1)$ and $f(x)$ is decreasing on $(0, \frac{1}{4}) \cup (1, \infty)$

- Find and classify critical numbers of $f(x)$.

  Solution:
  $f(x)$ is differentiable and $f'(x) = 0$ has two solutions $x = \frac{1}{4}, 1$. Hence $f(x)$ has two critical numbers $x = \frac{1}{4}, 1$. Since $f'(x) < 0$ for $x \in (0, \frac{1}{4})$ and $f'(x) > 0$ for $x \in (\frac{1}{4}, 1)$, $f(\frac{1}{4})$ is a local minimum.
  Since $f'(x) > 0$ for $x \in (\frac{1}{4}, 1)$ and $f'(x) < 0$ for $x \in (1, \infty)$, $f(1)$ is a local maximum.

- Find absolute extreme values of $f(x)$ on $[\frac{1}{2}, e]$.

  Solution:
  In the interval $[\frac{1}{2}, e]$, there is only one critical number $x = 1$. Hence the candidates for absolute extreme values are $f(1) = 3$, $f(\frac{1}{2}) = 2 + \ln 2$, and $f(e) = -2e^2 + 5e - 1$.
  Since $f(1) > f(\frac{1}{2}) > f(e)$, the absolute maximum value is $f(1)$, and the absolute minimum value is $f(e)$.

- Compute $f''(x)$. Find the intervals of concavity and inflection point(s).

  Solution:
  $f''(x) = -4 + \frac{1}{x^2}$.
  $f''(x) > 0$ for $x \in (0, \frac{1}{2})$. Hence $y = f(x)$ is concave upward on $(0, \frac{1}{2})$. $f''(x) < 0$ for $x \in (\frac{1}{2}, \infty)$. Hence $y = f(x)$ is concave downward on $(\frac{1}{2}, \infty)$. And $(\frac{1}{2}, f(\frac{1}{2}))$ is the inflection point.

3. Show that the equation $2x + \tan^{-1} x - 1 = 0$ has exactly one real root.
   Solution:
   Let $f(x) = 2x + \tan^{-1} x - 1$. $f(x)$ is continuous and differentiable on R. Since $f(0) = -1 < 0$ and $f(1) = 1 + \tan^{-1} 1 > 0$, there is a root for $f(x) = 0$ in the interval $(0, 1)$ by the intermediate value theorem.

   If there is another root for $f(x) = 0$, then by Rolle's theorem there is some point c between two roots such that $f'(c) = 0$.
   However, $f'(c) = 2 + \frac{1}{1 + c^2} > 2$. We obtain a contradiction. Hence there is only one root for $f(x) = 0$.